EXISTENCE RESULTS FOR QUASILINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS VIA TOPOLOGICAL METHODS

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Abstract. In this paper, existence and localization results of C^1-solutions to elliptic Dirichlet boundary value problems are established. The approach is based on the nonlinear alternative of Leray-Schauder.

1. Introduction

In this paper, we consider the boundary value problem

$$\begin{aligned} -\Delta_p u &= f(x,u), \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqq 1)$ is a nonempty bounded open set with smooth boundary $\partial\Omega$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function. We seek C^1-solutions, i.e. function $u \in C^1(\overline{\Omega})$ which satisfy (1) in the sense of distributions.

In recent years, many authors have studied the existence of solutions for problem (1) from several points of view and with different approaches (see,for example, [A, AR, CCN, CTY]). For instance, Afrouzi and Rasouli [AR] ensure the existence of solutions for special types of nonlinearities, by using the method of sub- and supersolutions.

Existence and multiplicity results for problem (1) are also presented by Anello [A], where f admits the decomposition $f = g + h$ with g and h two Carathéodory functions having no growth conditions with respect to the second variable. His approach is variational and mainly based on a critical point theorem by B. Ricceri.

In [CCN], Castro, Cossio and Neuberger apply the minmax principle to obtain sign-changing solutions for superlinear and asymptotically linear Dirichlet problems.

A novel variational approach is presented by Costa, Tehrani and Yang [CTY] to the question of existence and multiplicity of positive solutions to problem (1), where they consider both the sublinear and superlinear cases. Another useful method for the investigation of solutions to semilinear problems is based on the Leray-Schauder continuation principle, or equivalently, on Schaefers fixed point theorem. For example, in [GT] this method is used for solutions in Hölder spaces, while in [OP], solutions are found in Sobolev spaces.

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In this paper, we present new existence and localization results for C^1-solutions to problem (1), under suitable conditions on the nonlinearity f. No growth conditions of subcritical exponent type are required. Our approach is based on regularity results for the solutions of linear Dirichlet problems and again on the nonlinear alternative of Leray-Schauder (see [DG, Z]). We also notice that our present paper is motivated by the paper [MP] where the same results are obtained for semilinear elliptic boundary value problems.

Our approach is mainly based on the following well-known theorem.

Theorem 1. *Let $B[0,r]$ denote the closed ball in a Banach space E with radius r, and let $T : B[0,r] \to E$ be a compact operator. Then either*

(i) *the equation $\lambda Tu = u$ has a solution in $B[0,r]$ for $\lambda = 1$, or*
(ii) *there exists an element $u \in E$ with $\|u\| = r$ satisfying $\lambda Tu = u$ for some $0 < \lambda < 1$.*

It is worth noticing that contrary to most papers in the literature where the Leray-Schauder principle is used together with the a priori bounds technique, in the proof of our main result, Theorem 2, no a priori bounds of solutions of (3) are established. In addition, Theorem 2 not only that guarantees the existence of a solution, but also gives information about its localization. This is derived from a very general growth condition, inequality (3), which in particular contains both sublinear and superlinear cases without any restriction of exponent.

2. Main results

Here and in the sequel E will denote the space

$$C_0\left(\overline{\Omega}\right) = \left\{u \in C\left(\overline{\Omega}\right) : u = 0 \text{ on } \partial\Omega\right\}$$

endowed with the sup-norm

$$\|u\|_0 = \sup_{x\in\overline{\Omega}} |u(x)| .$$

Also by $C_0^1(\Omega)$ we mean the space $C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$. We start with an existence and localization principle for (1)

Theorem 2. *Assume that there is a constant $r > 0$, independent of $\lambda > 0$, with*

$$\|u\|_0 \neq r, \tag{2}$$

for any solution $u \in C_0^1(\overline{\Omega})$ to

$$\begin{aligned} -\Delta_p u &= \lambda f(x,u), \quad \text{in } \Omega,\\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{3}$$

and for each $\lambda \in (0,1)$. Then the boundary value problem (1) has at least one solution $u \in C_0^1(\overline{\Omega})$ with $\|u\|_0 \leqq r$.

In order to prove Theorem 2, we firstly recall a well-known property of the operator $-\Delta_p$.

Lemma 1 (See [AC], Lemma 1.1)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{1,\beta}$ for some $\beta \in (0,1)$ and $g \in L^\infty(\Omega)$. Then the problem*

$$\begin{aligned} &\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx = \int_\Omega g\varphi dx, \quad \text{for all } \varphi \in C_0^\infty(\Omega),\\ &u \in W_0^{1,p}(\Omega), \quad p > 1, \end{aligned} \tag{4}$$

has a unique solution $u \in C_0^1(\overline{\Omega})$. Moreover, if we define the operator $K : L^\infty(\Omega) \to C_0^1(\overline{\Omega}) : g \mapsto u$ where u is the unique solution of (6), then K is continuous, compact and order-preserving.

PROOF OF THEOREM 2. According to Lemma 1, the operator $(-\Delta_p)^{-1}$ from $L^\infty(\Omega)$ to $C_0^1(\overline{\Omega})$ is well-defined, continuous, compact and order-preserving. We shall apply Theorem 1 to $E = C_0(\overline{\Omega})$ and to the operator $T : C_0(\overline{\Omega}) \to C_0(\overline{\Omega})$, with $Tu = (-\Delta_p)^{-1}Fu$, where $F : C(\overline{\Omega}) \to C(\overline{\Omega})$ is given by $(Fu)(x) = f(x, u(x))$. Notice that, On the other hand, it is clear that the fixed points of T are the solutions of problem (1). Now the conclusion follows from Theorem 1 since condition (ii) is excluded by hypothesis. □

Theorem 2.1 immediately yields the following existence and localization result.

Theorem 3. *Assume that there exist nonnegative continuous functions α, β and a continuous nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$|f(x,u)| \leqq \alpha(x)\,\psi(|u|) + \beta(x), \quad \forall (x,u) \in \Omega \times \mathbb{R}. \tag{5}$$

Suppose in addition that there exists a real number $r > 0$ such that

$$r \geqq \left\| (-\Delta_p)^{-1} \alpha \right\|_0 \psi(r) + \left\| (-\Delta_p)^{-1} \beta \right\|_0. \tag{6}$$

Then the boundary value problem (1) *has at least one solution in $C_0^1(\overline{\Omega})$ with $\|u\|_0 \leqq r$.*

Proof. In order to apply Theorem 2, we have to show that condition (2) holds true for all solutions to (3). Assume u is any solution of (3) for some $\lambda \in (0,1)$ with $\|u\|_0 = r$. Then

$$u = \lambda T u = \lambda (-\Delta_p)^{-1} Fu.$$

Futhermore, for all $x \in \overline{\Omega}$, we have

$$\begin{aligned}
|u(x)| &= \lambda \left| (-\Delta_p)^{-1} Fu(x) \right| \\
&\leqq \lambda \left| (-\Delta_p)^{-1} \left(\alpha(x)\,\psi(|u(x)|) + \beta(x)\right) \right| \\
&\leqq \lambda \left(\left\| (-\Delta_p)^{-1} \alpha \right\|_0 \psi(\|u\|_0) + \left\| (-\Delta_p)^{-1} \beta \right\|_0 \right) \\
&\leqq \lambda \left(\left\| (-\Delta_p)^{-1} \alpha \right\|_0 \psi(r) + \left\| (-\Delta_p)^{-1} \beta \right\|_0 \right).
\end{aligned}$$

Taking the supermum in the above inequality, we obtain

$$\|u\|_0 \leqq \lambda \left(\left\| (-\Delta_p)^{-1} \alpha \right\|_0 \psi(r) + \left\| (-\Delta_p)^{-1} \beta \right\|_0 \right).$$

Therefore $r \leqq \lambda r < r$ since $\lambda \in (0,1)$ and $\|u\|_0 \leqq r$. This is a contradiction. □

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References

[A] G. Anello, Existence of solutions for a perturbed Dirichlet problem without growth conditions, *J. Math. Anal. Appl.* **330**(2), 1169-1178.

[AC] C. Azizieh and Ph. Clément, A priori estimates and continuation methods for positive solutions of p-Laplace equations, *J. Diff Eqns*, **179** (2002), 213-245.

[AR] G.A. Afrouzi and S.H. Rasouli, On positive solutions for some nonlinear semipositone elliptic boundary value problems *Nonlinear Analysis: Modelling and Control*, **11** (4), (2006), 323-329.

[CCN] A. Castro, J. Cossio and J.M. Neuberger, A minmax principle, index of the critical point, and existence of sign-changing solutions to elliptic boundary value problems, *Electron. J. Diff Eqns*, 1998, no. 2, 18 pp.

[CTY] D.G. Costa, H. Tehrani and J.J. Yang, On a variational approach to existence and multiplicity results for semipositone problems , *Electron. J. Diff Eqns*, 2006, no. 11, 10 pp.

[DG] J. Dugundji and A. Granas, *Fixed Point Theory, Monographie Math.*, Warsaw, 1982.

[GT] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 1983.

[MP] T. Moussaoui and R. Precup, Existence results for semilinear elliptic boundary value problems via topological methods, *Applied Mathematics Letters (2008)*, doi:10.1016/j.aml.2008.03.002.

[OP] D. ORegan and R. Precup, *Theorems of Leray-Schauder Type and Applications*, Gordon and Breach, Amsterdam, 2001.

[Z] E. Zeidler, *Nonlinear Functional Analysis : Part I*, Springer-Verlag, New York, 1985.

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